SELF-CONSISTENT BUNCHES OF OSCILLATING HEAVY PARTICLES IN AN ACCELERATOR

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The possibility of the existence of self-consistent spheroidal bunches having a constant density which consists of heavy particles or bodies is demonstrated. In a projection onto the transverse plane the particles describe circles or ellipses, while along the spheroid axis the particles oscillate sinusoidally. It is possible for an infinite set of particle distributions to appear with respect to the half-axes of the transverse ellipses and with respect to the amplitudes of the longitudinal oscillations.

The problem is considered of the stationary uniform filling of an ellipsoid of rotation (spheroid) with particles which perform sinusoidal oscillations

$$x = A_x \sin(\Omega_r t + \theta_x), \quad y = A_y \sin(\Omega_r t + \theta_y), \quad z = A_z \sin(\Omega_z t + \theta_z) \tag{1}$$

having stipulated angular frequencies $\Omega_x = \Omega_y = \Omega_r$ and Ω_z, in a projection onto the spheroid axis.

This problem arose in considering bunches of charged elementary particles (protons) in a linear accelerator [1]. In a spheroidal proton bunch its intrinsic electric field summed with the accelerating and focusing fields of the accelerator has components which are proportional to identical coordinates and are independent of other coordinates:

$$E_x = -K_r x, \quad D_y = -K_r y, \quad E_z = -K_z z \tag{2}$$

For self-consistency of the bunch it is necessary that the particles which perform oscillations of the form (1) in the field (2) be so distributed with respect to the amplitudes and phases of the oscillations A, θ (with respect to the integrals of motion) that at any instant they fill the spheroid with a stipulated constant density, since only under this condition will the field have the form (2).

Equations (2) likewise express the gravitational field inside a homogeneous spheroidal bunch of heavy particles or bodies. Under these conditions

$$\Omega_r^2 = K_r = 4\pi\sigma G M_r, \quad \Omega_z^2 = K_z = 4\pi\sigma G M_z$$

where G is the gravitational constant; σ is the density of the bunch; M_r and M_z are coefficients which depend on the shape of the spheroid ($2M_r + M_z = 1$). Thus, the specific physics content of the problem of self-consistent bunches may be different. These may be either clouds of cosmic dust, clusters of stars, or bunches of protons in an accelerator.

It is assumed that the relativistic effects, fluctuations of the field (2), and collisions and exchange of energy between particles may be neglected. The problem of the stability of the bunches is not considered. Heavy particles, of course, may have identical masses.

A spheroid having the half-axes $a_x = a_y = a_r$ and a_z can be described by the equation

$$\frac{x^2 + y^2}{a_r^2} + \frac{z^2}{a_z^2} = 1 \tag{3}$$

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Let us isolate particles having a certain fixed amplitude A_z of the longitudinal oscillations from the bunch. These particles are confined in a cylinder of length $2A_z$ and radius $a_r\sqrt{1-(A_z/a_z)^2}$ inscribed in the spheroid (3). The fraction of the period of the longitudinal oscillations $T_z = 2\pi/\Omega_z$ during which the particle is situated in the interval from z to z + dz is

$$\frac{2dz}{T_z dz/dt} = \frac{2dz}{T_z \Omega_z A_z \cos(\Omega_z t + \theta_z)} = \frac{dz}{\pi\sqrt{A_z^2 - z^2}}$$

In view of the stationarity of the bunch the particle distribution over the phases θ_z must be uniform. Then the particle density having a given amplitude A_z can be expressed with allowance for axial symmetry of the bunch in the form

$$\frac{N(r, A_z)}{\pi\sqrt{A_z^2 - z^2}} \qquad (r = \sqrt{x^2 + y^2}) \tag{4}$$

where N is a certain function of A_z and of the radial coordinate r.

The total particle density σ = const at the point r, z can be expressed by an integral of the partial densities (4) over all the amplitudes A_z which are not less than z and do not cause the particle to depart beyond the limits of the spheroid for a given r:

$$\sigma = \frac{1}{\pi}\int_z^{A_{zm}} \frac{N(r, A_z)\, dA_z}{\sqrt{A_z^2 - z^2}}, \qquad A_{zm} = a_z\sqrt{1 - r^2/a_r^2} \tag{5}$$

From this integral equation, which can easily be reduced to the Abel equation, we find

$$N(r, A_z) = \frac{2\sigma A_z}{\sqrt{a_z^2(1 - r^2/a_r^2) - A_z^2}} = \frac{CA_z}{\pi^2\sqrt{A_r^2 - r^2}}, \quad A_r = a_r\sqrt{1 - A_z^2/a_z^2}, \quad C = 2\pi^2\sigma a_r/a_z \tag{6}$$

The initial three-dimensional problem has now been reduced to a two-dimensional problem. It is required to find the distribution of the particles with respect to the amplitudes and phases of the transverse oscillations which has a given amplitude A_z and for which the dependence of particle density on the radial coordinate r can be described by the function (6). Let us consider solutions of this problem.

The First Solution. It consists simply in the motion of particles along circles r = R in the projection onto the transverse plane. Such motion holds for the conditions that for each particle $A_x = A_y = R$, $|\theta_x - \theta_y| = \pi/2$, and the particles are distributed uniformly with respect to the oscillation phases θ_x (and consequently with respect to θ_y). Of course, all of the particles rotate at an identical frequency Ω_z. Under these conditions the particle density can be expressed by the function N given in (6) as a function of the amplitudes (rotation radii) r = R, while the density of the particle distribution with respect to the radii r = R is expressed by the function $2\pi rN$.

However, other less trivial distributions are likewise possible. The transverse direction of the particle trajectory in accordance with (1) is an ellipse with its center x = y = 0 and the half-axes

$$a, b = \left(\sqrt{A_x^2 + A_y^2 + 2A_xA_y \sin|\theta_x - \theta_y|} \pm \sqrt{A_x^2 + A_y^2 + 2A_xA_y \sin|\theta_x - \theta_y|}\right)/2 \tag{7}$$

in the general case, the major axis of the ellipse making an angle

$$\eta = \frac{1}{2}\,\mathrm{arc\,tg}\left[\frac{2A_xA_y}{A_x^2 - A_y^2}\cos(\theta_x - \theta_y)\right]$$

with the x axis.

Instead of the particle distribution with respect to the amplitudes and phases A_x, A_y, θ_x, θ_y it is more convenient to seek the distribution with respect to the parameters a, b, and η of the ellipses and with respect to the phases of motion along the ellipses. In view of the axial symmetry and stationarity of the bunch the particle distribution with respect to the orientation angles η of the ellipses and the phases of the motion along the ellipses must be uniform. The fraction $T_r = 2\pi/\Omega_r$ of the period of motion along the ellipse during which the particle is in a cylindrical layer extending from r to r + dr is equal to

$$\frac{4dr}{T_r dr/dt} = \frac{2r dr}{\pi\sqrt{(a^2 - r^2)(r^2 - b^2)}}$$

in accordance with Eqs. (1) and (7).

The distribution density of particles having given a and b is $2\pi r$ times as small and can be expressed in the form

$$\frac{M(a, b, A_z)}{\pi^2 \sqrt{(a^2 - r^2)(r^2 - b^2)}} \tag{8}$$

Here M is the desired distribution function of the particles with respect to the parameters a, b, A_z. The distribution density of particles having a given A_z (6) for a given r is expressed by the integral of the partial densities (8) over all b which do not exceed r and over all a which are not less than r and not larger than A_r:

$$\frac{1}{\pi^2} \int_r^{A_r} da \int_0^r \frac{M(a, b, A_z)\, db}{\sqrt{(a^2 - r^2)(r^2 - b^2)}} = \frac{CA_z}{\pi^2 \sqrt{A_r^2 - r^2}} \tag{9}$$

We shall seek the function M in the form of the product of two functions:

$$M(a, b, A_z) = f(a, A_z)\, \varphi(b, A_z)$$

The integral equation (9) then decomposes into two equations:

$$\int_0^r \frac{\varphi(b, A_z)\, db}{\sqrt{r^2 - b^2}} = \Phi(r, A_z) \tag{10}$$

$$\int_r^{A_r} \frac{f(a, A_z)\, da}{\sqrt{a^2 - r^2}} = \frac{CA_z}{\Phi(r, A_z)\sqrt{A_r^2 - r^2}} \tag{11}$$

If the function φ is specified, then we find Φ from (10), and it remains for us to find f from the integral equation (11). However, if we specify f, then we find Φ from (11) and it will be necessary to find φ from Eq. (10). Of course, one may also specify the function Φ, and then it remains for us to find φ and f from the integral equations (10) and (11). Reducing each of them to an Abel equation, we obtain

$$\varphi(b, A_z) = \frac{2}{\pi} \frac{d}{db} \int_0^b \frac{\Phi(r, A_z)\, r\,dr}{\sqrt{b^2 - r^2}} \tag{12}$$

$$f(a, A_z) = \frac{CA_z}{\pi} \left[\frac{\pi}{A_r \Phi(A_r, A_z)} \delta\left(\frac{a}{A_z} - 1\right) - \frac{d}{da} \int_a^{A_r} \frac{2r\,dr}{\Phi(r, A_z)\sqrt{(A_r^2 - r^2)(r^2 - a^2)}} \right] \tag{13}$$

Equation (13) includes a δ-function. The functions f, φ, $\Phi^{\pm 1}$ must be integrable and nonnegative, the relationships

$$\Phi(A_r, A_z) \neq 0, \quad \Phi(a, A_z) \neq 0, \quad \Phi(b, A_z) \neq \infty$$

being valid.

The distribution $M = f\varphi$ which has been found satisfies the normalization

$$\int_0^{a_z} dA_z \int_0^{A_r} da \int_0^a M(a, b, A_z)\, db = \frac{4}{3} \pi a_r^2 a_z \sigma \tag{14}$$

The mechanical (or magnetic) moments of the bunches depend on the distribution of the particles with respect to their directions of rotation about the axis and their proper angular momenta.

Since one of the functions f, φ, Φ may be chosen arbitrarily, the problem has an infinite set of solutions. In addition to the first solution presented above (r = R), one can also obtain other solutions from Eqs. (10)-(13), as well as families of solutions. Let us present examples.

The Second Solution. Assume $\varphi = 1$. From Eqs. (10) and (13) we find

$$\Phi = \frac{\pi}{2}, \quad M = f = \frac{4\pi\sigma a_r}{a_z} \frac{A_z}{A_r} \delta\left(\frac{a}{A_r} - 1\right)$$

i.e., $a = A_r$. The first and second solutions were obtained in [1].

The Third Solution. Assume $f = a\,(A_r^2 - a^2)^{-\varepsilon}$, $0 \leq \varepsilon < 1$. From Eqs. (11) and (12) we obtain

$$\Phi = \frac{CA_z}{\psi(\varepsilon)(A_r^2 - r^2)^{1-\varepsilon}}, \quad \psi(\varepsilon) = \int_0^1 \frac{d\tau}{(1 - \tau^2)^{\varepsilon}}$$

$$M = \frac{4\pi\sigma a_r}{\psi(\varepsilon) a_z} \frac{A_z a}{(A_r^2 - a^2)^\varepsilon} \left[\frac{A_r^{2\varepsilon}}{A_r^2 - b^2} + \frac{(1-2\varepsilon) b}{(A_r^2 - b^2)^{3/2-\varepsilon}} \int_0^B \frac{d\tau}{(1+\tau^2)^{1-\varepsilon}} \right],$$

$$B = \frac{a}{\sqrt{A_r^2 - b^2}}$$

For $\varepsilon = 0, 1/2$ we obtain $\psi(\varepsilon) = 1, \pi/2$, and the expression for M is simplified.

The Fourth Solution. Assume $\Phi = r^n$, $n > 0$. From (12) we find $\varphi = \chi(n) b^n$. For even n the integrals in (12) and (13) can be calculated in finite form. Thus, for $\Phi = r^2$ we have

$$\varphi = \frac{4b^2}{\pi}, \quad M = \frac{4b^2}{\pi} f = \frac{8\pi\sigma a_r}{a_z} \frac{A_z b^2}{A_r} \left[\frac{1}{a^2} + \frac{1}{A_r^2} \delta\left(\frac{a}{A_r} - 1\right)\right]$$

Linear combinations of particular solutions which satisfy the normalization (14) and are not negative are likewise solutions of the problem.

Thus, we have demonstrated the possibility that self-consistent bunches of oscillating heavy particles exist. The bunches have the form of constant-density spheroids. In projection onto the transverse plane the particles describe circles or ellipses, while they perform sinusoidal oscillations in the direction of the spheroid axis. It is possible to have an infinite set of different particle distributions with respect to the semiaxes of the transverse ellipses and with respect to the amplitudes of the longitudinal oscillations.

LITERATURE CITED

1. B. I. Bondarev and A. D. Vlasov, "On the self-consistent distribution of particles and limiting current in a linear accelerator," Atom. Énerg., 19, No. 5, 423 (1965).